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# Fully coupled resonant-triad interactions in a free shear layer

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We report the results of an investigation of the weakly nonlinear evolution of a triad of waves, each slightly amplified on a linear basis, that are superimposed on a $\tanh y$ mixing layer. The triad consists of a plane wave and a pair of oblique modes that act as a subharmonic of order 1/2. The oblique modes are inclined at approximately $\pm 60°$ to the mean flow direction and because the resonance conditions are satisfied exactly the analysis is entirely self-consistent as an asymptotic theory. The nonlinearity first occurs within the critical layer and the initial interaction is of the parametric resonance type. This produces faster than exponential growth of the oblique waves, behaviour observed recently in the experiments of Corke & Kusek (1993). The critical-layer dynamics lead subsequently to coupled integro-differential equations governing the amplitude evolution and, as first shown in related work by Goldstein & Lee (1992) on boundary layers in an adverse pressure gradient, these equations develop singularities in a finite time.

## 1. Introduction

Experiments on the transition to turbulence of incompressible mixing layers that involve external forcing show that this process occurs in several stages. When the forcing is two-dimensional, the initial instability is primarily two-dimensional. This observation is consistent with linear stability theory, where plane waves propagating in the mean flow direction are known to have the largest growth rates. Nonlinearity results in eventual saturation of the most amplified plane wave and, as it tends toward equilibration, a subharmonic wave having one-half the frequency of the dominant initial disturbance makes its appearance (as do higher harmonics). This intriguing phenomenon manifests itself in smoke visualization experiments as vortex pairing (see, e.g. Freymuth 1966) and is also apparent from hot-wire signals (figure 8 in Sato 1959 provides a striking example).

The survey article by Ho & Huerre (1984) reviews numerous experiments, several theoretical descriptions and discusses the technological implications of vortex pairing. The same article also cites some of the first observations of three-dimensional structures which often take the form of counter-rotating pairs of vortices whose axes are in the direction of the mean flow. Considerable experimental work has been carried out more recently on these three-dimensional structures. Whereas pairing was long believed to be essentially a two-dimensional process, the more recent studies, as discussed by Dallard & Browand (1993), indicate that pairing is three-dimensional. This seems to be true for both laminar and turbulent mixing layers whether or not the flow is forced. Of particular interest relative to the present investigation are the experiments reported by Nygaard & Glezer (1991) in which it was found that streamwise vortices could be generated that sometimes formed upstream of the initial pairing of the primary spanwise vortices. The sequence of events during transition, in other words, is quite sensitive to initial conditions, particularly the nature of any forcing upstream of the initial instability. The latter article also includes an extensive review of the more recent experiments on three-dimensional aspects of mixing-layer transition.

The present paper is concerned with the analysis of resonant interactions believed to play a significant role in the phenomena described above. An important contribution toward that end is contained in ideas first put forward by Kelly (1967) who showed that certain features observed in the early stages of vortex pairing could be modelled (and in some cases predicted) by studying the resonant interaction of two disturbances whose wavenumbers are in the ratio 2:1. The shorter wave in Kelly's analysis was taken to be periodic in space and time (i.e. neutral) whereas the long wave, termed a subharmonic, had, initially, a much smaller amplitude. Its instability, due to both linear and parametric resonance effects, corresponds to the onset of pairing. Subsequently, Patnaik, Sherman & Corcos (1976) performed numerical simulations of instability in a stratified mixing layer and showed that the entire pairing process could be modelled *in the temporal case* by following the evolution of two interacting Fourier modes with wavenumbers in the same 2:1 ratio.

Kelly, however, pointed out a number of discrepancies between his subharmonic resonance model and the experimental events it is intended to describe. For example, the dispersive character of spatially growing plane waves detunes the resonance to the point that it would not be operational in a situation where (i) the shear flow is of mixing-layer form and truly parallel; and (ii) the fundamental (shorter wave) disturbance is the fastest growing wave of linear theory. Yet, despite this and other difficulties, there is sufficient agreement between theory and experiment to leave little doubt that the basic mechanism is correct. We will present below an analysis that retains this essential mechanism, but puts the mathematics on a more rational footing by utilizing a pair of oblique modes as the subharmonic. First, however, we comment briefly on two extensions of the subharmonic resonance approach that have appeared since the original work by Kelly.

First, Pierrehumbert & Widnall (1982) have studied the instability of Stuart vortices to perturbations including a spanwise component. Their analysis is similar to that of Kelly (and different from our own) in that the subharmonic is viewed as a secondary instability of a flow that is spatially periodic in the flow direction. The approach of Monkewitz (1988), on the other hand, allows both the fundamental and subharmonic perturbations to be of the same order of magnitude. The fundamental is taken essentially to be neutral and it is found that the exponential amplification of the subharmonic is enhanced owing to the resonance. The effect of a slight detuning of the frequencies is also taken into account and discussed relative to experimental observations. Because of the exponential amplification of the subharmonic and the presence of a 'detuning parameter' which cannot vanish in the spatial case if the interacting modes are both two-dimensional, the perturbation expansion employed by Monkewitz is restricted to only a limited distance in the flow direction. However, while briefly considering a subharmonic comprising a pair of slightly oblique modes, he notes that for a particular spanwise wavenumber (estimated to be 0.35) the detuning parameter vanishes and so the resonance would be 'enhanced'.

The foregoing observation is directly related to our own approach (although it will be seen that the special spanwise wavenumber does not correspond to an amplified mode, as assumed by Monkewitz, and consequently is more than double the above estimate). We consider as our basic state a tanh $y$ shear layer upon which is superimposed a resonant triad of neutral modes comprising a plane wave and a pair of oblique modes, each inclined at 60° with respect to the mean flow direction. Because the resonance conditions, as will be shown in §2, are satisfied exactly our solution will remain valid in the limit as the small parameter in the analysis goes to zero. Unlike previous analyses of the phenomenon described above, our results will therefore be correct in a strict asymptotic sense. Experience suggests that when this is the case, there is reason to hope that the results will be qualitatively correct even when we are considering linearly amplified waves and the perturbation is not particularly small.

As the basic small parameter in the analysis, it is most appropriate to choose the departure of the prescribed wavenumber $\alpha$ of the plane wave from its neutral value of 1. We, in effect, will do this but for convenience in interpreting the final results it is helpful to introduce an $O(1)$ constant which we denote $\alpha_1$ and write

$$\alpha = 1-\epsilon^{1/3}\alpha_1, \quad \alpha_1 > 0, \tag{1.1}$$

where $\epsilon$ is an amplitude parameter. It is necessary that $1-\alpha$ be positive and, preferably, not too small because we will employ a non-equilibrium critical layer in which the flow evolves on a relatively fast timescale, namely, $T = \epsilon^{1/3}t$ (or $X = \epsilon^{1/3}x$ in the spatial case). This means that our results are more applicable to the high-Reynolds-number experiments of Nygaard & Glezer than to experiments conducted at comparatively low Reynolds numbers, such as those reported by Lasheras, Cho & Maxworthy (1986). A viscous critical layer would be more appropriate in the latter case; however, as discussed by Churilov & Shukhman (1987) in some detail, consistency would then require a proper accounting of the basic flow's diffusion.

For a tanh $y$ shear layer, Lin's perturbation formula can be utilized to estimate the linear amplification rate for a near-neutral disturbance. Using this estimate, Churilov & Shukhman note that one condition for an inviscid non-equilibrium critical layer to be appropriate is satisfaction of the inequality $Re^{-1/3} \ll 2(1-\alpha)/\pi$, where $Re^{-1/3}$ is the viscous critical-layer thickness and $Re$ is the Reynolds number. In order that non-equilibrium effects dominate over nonlinearity in the critical layer, the analogous condition is $\epsilon^{1/2} \ll 2(1-\alpha)/\pi$. Satisfaction of these inequalities is crucial because, as is particularly true in the non-equilibrium approach, the critical-layer dynamics govern the evolution of the flow well into the nonlinear régime. Extension of these arguments to the case of perturbations amplifying in both space and time can be found in §5 of Huerre (1987). For a review of various critical-layer theories, especially for nonlinear and viscous critical layers, the reader is referred to the survey article by Maslowe (1986).

A brief explanation for the ordering of the perturbation amplitudes is called for because we will choose the asymptotic scaling of the oblique modes to be slightly larger than the plane wave, $O(\epsilon)$ as compared with $O(\epsilon^{4/3})$. This choice, unless there is strong forcing of the oblique modes, is inappropriate for the initial stages of the interaction. However, as discussed in a recent survey article by Goldstein (1994), the sort of analysis presented here describes an intermediate stage of the evolution in which the oblique modes have now grown larger than the plane wave owing to a parametric resonance which seems to occur in a number of shear flows. For a tanh $y$ shear layer, analysis of the earlier stage in which all modes are of $O(\epsilon)$ was found by Mallier & Maslowe (1994) to lead to the amplitude equations

$$\frac{\mathrm{d}A_{20}}{\mathrm{d}\tau}-\frac{2\alpha_1}{\pi}A_{20} = 0 \tag{1.2}$$

and
$$\frac{\mathrm{d}A_{11}}{\mathrm{d}\tau}-\frac{\alpha_1}{\pi}A_{11}=\frac{3\mathrm{i}}{16}\int_0^\infty \tau_0^2 A_{20}(\tau-\tau_0)\,A_{11}^*(\tau-2\tau_0)\,\mathrm{d}\tau_0, \tag{1.3}$$

where $A_{20}$ and $A_{11}$ are the amplitudes of the plane and oblique waves, respectively, and $\tau = \epsilon^{1/4}t$. (The subscript assignments are explained in §2 and $\alpha_1$ is related to $\alpha$ as in (1.1), but with $\epsilon^{1/4}$ in place of $\epsilon^{1/3}$.)

Comparing (1.2) and (1.3) with conventional equations of resonant interaction theory (see e.g. Craik 1986) we would expect to find a nonlinear term involving an integral of $A_{11}^2$ on the right-hand side of (1.2). The coefficient of that term turns out to be zero here, as well as in the related studies discussed in the next paragraph. As a result, the plane wave continues to amplify as in linear theory, whereas the solution of (1.3) predicts very rapid (exponential of an exponential) amplification of the oblique waves for $\tau \gg 1$. Experimental observation of such behaviour in the thin mixing layer at the edge of a circular jet was reported very recently by Corke & Kusek (1993; see figures 6 and 12). While our theory differs from the approach of Monkewitz and Kelly in that there is no threshold amplitude for $A_{20}$, it still must be at least of the same order of magnitude as the oblique modes in order to appear in (1.3) and produce such rapid amplification of the latter.

Clearly, at some time beyond the parametric resonance stage, $t \sim O(\epsilon^{-1/4})$, there must be a so-called back-reaction of the oblique waves on the plane wave and the following stage will be governed by coupled evolution equations for $A_{20}$ and $A_{11}$. Goldstein & Lee (1992) have identified a distinguished limit that leads to evolution equations that are strongly coupled in their investigation of long-wave interactions in an adverse-pressure-gradient boundary layer. Because in the present study the relevant wavenumbers are $O(1)$, our scaling differs from that of Goldstein & Lee, but is identical with the analysis by Wu (1992) of the Stokes layer. It will be seen that our treatment of the free shear layer is simplified in comparison with the aforementioned examples thanks to the availability of a closed-form neutral solution for the tanh $y$ mixing layer. We are, as a result, able to determine explicitly the coefficients of all terms in our amplitude equations without the need for numerical computation or long-wave expansions. The choice of scales, incidentally, is subtle and depends, for example, on the order at which the first velocity jump across the critical layer occurs.

The end result of our analysis will be two fully coupled nonlinear integro-differential equations for the amplitudes, each of which develops a singularity in a finite time (or distance). A similar singularity appears even without a plane wave as was shown by Goldstein & Choi (1989) who considered a disturbance of two oblique waves on a mixing-layer profile. The actual significance of this 'explosive instability' remains to be established and is a subject of current research. Finally, we note that both Goldstein & Lee (1992) and Wu (1992) have shown that the parametric resonant stage described by (1.2) and (1.3) is recovered from the fully coupled equations in the limit $A_{11} \ll A_{20}$, so the present scaling appears to be more general.

Before giving the derivation of these equations, we point out another significant feature of the analysis, namely that the interaction of the oblique waves in the critical layer generates a mean streamwise vortex motion outside the critical layer. The velocity component in the flow direction turns out to be as large, i.e. $O(\epsilon)$, as the oblique waves that induce it, whereas the other two components are slightly smaller, specifically $O(\epsilon^{4/3})$.

In the following section, we present the basic perturbation expansion that is substituted into the Euler equations to describe the motion outside the critical layer. Then, in §3, the analysis of the non-equilibrium critical layer is presented. The problem is formulated initially using the temporal theory because, for reasons of economy, most numerical simulations are done that way. In particular, some comparisons will be made in §4 with the recent Navier–Stokes computations of Schoppe, Hussain & Metcalfe (1994) which, in some important respects, agree with our theory. Finally, in the same section, the amplitude equations for spatial evolution are discussed relative to experimental observations.

## 2. Formulation and outer expansion

We consider the stability of the dimensionless mixing-layer profile

$$\bar{u}(y) = u_m + \tanh y \tag{2.1}$$

by adding a small perturbation of $O(\epsilon)$, where $\epsilon \ll 1$ is a dimensionless amplitude parameter. The equations of motion can be written

$$\frac{\partial \boldsymbol{q}}{\partial t} + (\boldsymbol{q}\cdot\boldsymbol{\nabla})\,\boldsymbol{q} = -\boldsymbol{\nabla} p \tag{2.2}$$

and

$$\boldsymbol{\nabla}\cdot\boldsymbol{q} = 0, \tag{2.3}$$

where we have supposed the fluid to be inviscid and incompressible. The temporal stability problem is independent of $u_m$, the mean velocity in (2.1), so we will set $u_m = 0$ until later in the paper when we discuss the spatial case.

The velocity components $\boldsymbol{q} = (\bar{u}+\epsilon u, \epsilon v, \epsilon w)$ and the perturbation pressure $\epsilon p$ are expanded as follows:

$$u = u^{(1)} + \epsilon^{1/3}u^{(2)} + \epsilon^{2/3}u^{(3)} + \dots, \tag{2.4}$$

$$v = v^{(1)} + \epsilon^{1/3}v^{(2)} + \epsilon^{2/3}v^{(3)} + \dots, \tag{2.5}$$

$$w = w^{(1)} + \epsilon^{1/3}w^{(2)} + \epsilon^{2/3}w^{(3)} + \dots \tag{2.6}$$

and

$$p = p^{(1)} + \epsilon^{1/3}p^{(2)} + \epsilon^{2/3}p^{(3)} + \dots. \tag{2.7}$$

In the linearized problem, perturbations proportional to $\exp[\mathrm{i}(\alpha x + \beta z - \alpha c t)]$ are considered and the $y$-dependent part of $v^{(1)}$, which we denote $\bar{v}_1$, satisfies the Rayleigh equation

$$\bar{v}_1'' - \bar{\alpha}^2\bar{v}_1 - \left(\frac{\bar{u}''}{\bar{u}-c}\right)\bar{v}_1 = 0, \tag{2.8}$$

where $\bar{\alpha} = (\alpha^2+\beta^2)^{1/2}$.

For the mixing-layer profile $\bar{u} = \tanh y$, Curle (1956) has found the following neutral solution of the eigenvalue problem consisting of (2.8) and the boundary conditions that $\bar{v}_1 \rightarrow 0$ as $y \rightarrow \pm\infty$:

$$\bar{v}_1 = \operatorname{sech} y, \quad \bar{\alpha}^2 = \alpha^2+\beta^2 = 1 \quad \text{and} \quad c = 0. \tag{2.9}$$

This solution is employed in our lowest-order disturbance which is a subharmonic of the plane wave. It comprises two oblique modes of equal amplitude, equally inclined to the mean flow, so that in the spanwise direction we have a standing wave. The vertical velocity at lowest order is accordingly of the form

$$v^{(1)} = \{A_{11}(T)\,\mathrm{e}^{\mathrm{i}x/2} + A_{11}^{*}\,\mathrm{e}^{-\mathrm{i}x/2}\}\,\bar{v}_1(y)\,2\cos\beta_0 z, \tag{2.10}$$

where $\beta_0 = \sqrt{3}/2$, $\bar{v}_1$ is given by (2.9) and $T = \epsilon^{1/3}t$ is a slow timescale, i.e. the method of multiple scales will be employed. The factor of 2 is introduced because it will simplify the subsequent development to utilize complex exponentials along with the identity $2\cos\beta_0 z = \exp(i\beta_0 z) + \exp(-i\beta_0 z)$.

A triad of neutral modes satisfying exactly the conditions for resonance could be chosen according to (2.9) by including a plane wave with wavenumber $\alpha = 1$. However, as discussed in §1, we will depart slightly from this scheme by setting $\alpha < 1$ by an amount $O(\epsilon^{1/3})$, as indicated by (1.1), so that the plane wave is amplified on a linear basis and the critical layer is of the non-equilibrium type. A second departure, as discussed in §1, is that the plane wave has an amplitude smaller by a factor $\epsilon^{1/3}$, this scaling being appropriate to describe the fully coupled stage of evolution. With regard to the wavenumbers for the pair of oblique modes, in most of the paper we will suppose that $\beta/\alpha = \surd 3/2$, where $\alpha$ is one-half the plane wave $\alpha$ defined by (1.1); however, the more general situation where the angle of inclination of the oblique waves can be less than 60° will be discussed in connection with the spatial case.

An observation that we make, in passing, relative to (2.9) is that because of the non-dispersiveness of the solution, it exhibits the phenomenon of '$n$th harmonic resonance'. This term originates from studies of capillary–gravity waves (Wilton's ripples) and, to describe our approach, it is arguably more appropriate than 'subharmonic resonance'. However, we retain the latter terminology to facilitate comparison with previous articles on mixing layers. In any case, the solution (2.9) is resonant for $\alpha = 1/n$ and, consequently, an infinite number of resonant combinations are possible; however, the fastest timescale is associated with $n = 2$, the case we analyse, and this resonance should therefore be the most observable. None the less, the experiments of Ho & Huang (1982) demonstrate clearly that forcing at an appropriate frequency (see their figure 2) can generate some of the higher resonances.

Returning now to the present analysis, the procedure for determining the remaining terms in (2.4)–(2.7) will be outlined only briefly because the details in many respects parallel the investigation of Benney (1961). Both $u^{(1)}$ and $w^{(1)}$ contain terms having first-order poles at the critical point $y = 0$. A significant difference, however, due to our use of a non-equilibrium critical layer rather than the steady viscous critical layer employed by Benney is that $u^{(1)}$ contains an additional term and is of the form

$$u^{(1)} = \{A_{11}\bar{u}_1(y)\,e^{ix/2} + A_{11}^*\bar{u}_1^*\,e^{-ix/2}\}\,2\cos\beta_0 z + 2u_{02}(y,T)\cos 2\beta_0 z, \qquad (2.11)$$

where $\bar{u}_1$ contains a term proportional to $\operatorname{cosech} y$ (see Benney, §4) and the second term in (2.11) is a spanwise mean flow component induced by the critical layer. The need for such a term, representing a streamwise vortex motion, was first established by Goldstein & Choi (1989). For the $\tanh y$ mixing layer,

$$u_{02} = \pm\surd 3\, C_{02}^{(1)\pm}\, e^{\mp\surd 3 y}\operatorname{sech} y, \qquad (2.12)$$

where $\pm$ denotes above and below the critical layer.

The presence of singular terms such as $\bar{u}_1$ generates discontinuities in certain higher-order velocity components and these must be smoothed out by the critical-layer solution. Although the general form of the amplitude equation is determined by the critical layer, the values of coefficients appearing in those equations must be determined by matching to the outer expansion.

Returning now to the expansions (2.4)–(2.7), the $O(\epsilon^{1/3})$ terms include the plane wave, and $u$- and $v$-components of the mean streamwise vortex motion and perturbations to the oblique waves. Beginning with the latter, and introducing the notation $v_{lm}^{(n)}$ to mean the term at $O(\epsilon^{(n-1)/3})$ containing the factor $\exp\{i(lx+\surd 3\, mz)/2\}$, we find that the quantity $v_{11}^{(2)}$ satisfies a non-homogeneous Rayleigh equation which can be written

$$\mathscr{L}_1 v_{11}^{(2)} = -2\alpha_1 A_{11}\operatorname{sech} y - 4iA_{11}'\operatorname{sech}^2 y\operatorname{cosech} y, \qquad (2.13)$$

where $\mathscr{L}_1$ is the Rayleigh operator in (2.8) with $\bar{\alpha} = 1$ and $\alpha_1$, as defined by (1.1), indicates departure from the neutral wavenumber. Because both homogeneous solutions are known in closed form, the method of variation of parameters can be used to write the general solution of (2.13) as

$$\begin{aligned} v_{11}^{(2)} &= C_{11}^{(2)\pm} \operatorname{sech} y + D_{11}^{(2)\pm}(y \operatorname{sech} y + \sinh y) - \alpha_1 A_{11} \cosh y \\ &+ 2\mathrm{i}A'_{11}\left[2 \operatorname{sech} y \int_0^y y_1 \operatorname{cosech} 2y_1 \, \mathrm{d}y_1 - (y \operatorname{sech} y + \sinh y) \log(\tanh|y|)\right]. \end{aligned} \quad (2.14)$$

Imposing the homogeneous boundary conditions as $y \rightarrow \pm\infty$ shows that there is a jump in $D_{11}^{(2)}$ which is related to the amplitude of the oblique waves by

$$D_{11}^{(2)+} - D_{11}^{(2)-} = 2\alpha_1 A_{11}. \quad (2.15)$$

Equations (2.2) and (2.3) can now be used to solve for $u_{11}^{(2)}$, $w_{11}^{(2)}$ and $p_{11}^{(2)}$.

As was the case with the lowest-order oblique modes (2.10), the lowest-order plane wave employs the neutral solution given by (2.9) and the vertical velocity due to this term is therefore

$$v_{20}^{(2)} = \{A_{20}(T)\,\mathrm{e}^{\mathrm{i}x} + A_{20}^{*}\,\mathrm{e}^{-\mathrm{i}x}\}\,\bar{v}_1(y). \quad (2.16)$$

At $O(\epsilon^{2/3})$ in the expansions (2.4)–(2.7), we need only calculate the perturbation to the plane wave, and we find that the quantity $v_{20}^{(3)}$ obeys an equation similar to (2.13) for $v_{11}^{(2)}$, namely

$$\mathscr{L}_1 v_{20}^{(3)} = -2\alpha_1 A_{20} \operatorname{sech} y - 2\mathrm{i} A'_{20} \operatorname{sech}^2 y \operatorname{cosech} y, \quad (2.17)$$

which has a solution similar in form to (2.14), but with $C_{11}^{(2)\pm}$ and $D_{11}^{(2)\pm}$ replaced by $C_{20}^{(2)\pm}$ and $D_{20}^{(2)\pm}$, respectively, and $A_{11}$ and $A'_{11}$ replaced by $A_{20}$ and $\frac{1}{2}A'_{20}$. Imposing the homogeneous boundary conditions as $y \rightarrow \pm\infty$ leads to a jump in $D_{20}^{(2)}$ given by

$$D_{20}^{(2)+} - D_{20}^{(2)-} = 2\alpha_1 A_{20}. \quad (2.18)$$

It will be seen that matching the jumps in (2.15) and (2.18) to the critical-layer solution derived in the following section will lead to the amplitude equations governing the temporal evolution of $A_{11}$ and $A_{20}$. Before continuing, however, we will note one procedural difference compared with Hickernell (1984) and other papers employing non-equilibrium critical layers. Because an analytical solution was not available for the outer problem, the jump conditions in these papers were derived from a generalization of the usual adjoint orthogonality condition in which it is implicit that an outer solution can be found satisfying the boundary conditions. Here, on the other hand, we employ the procedure described in Benney & Maslowe (1975, §3) and actually find the outer solution. The two procedures should lead to the same result for the amplitude equations.

## 3. Critical-layer analysis

To obtain evolution equations for $A_{20}$ and $A_{11}$, we shall now pose an inner expansion in the critical layer, where the outer expansion becomes disordered, in order to obtain expressions for the jumps. To this end, we introduce the inner variables $Y = \epsilon^{-1/3}y$, $U = \epsilon^{-1/3}\bar{u} + \epsilon^{2/3}u$, $V = \epsilon^{1/3}v$, $W = \epsilon^{2/3}w$ and $P = \epsilon^{-1/3}p$ inside the critical layer, where it should be recalled that $\epsilon$ is the order of magnitude of the oblique wave disturbance in the outer expansion, so that the governing equations become

$$\left.\begin{aligned} U_T+UU_x+VU_Y+WU_z+\epsilon^{2/3}P_x &= 0,\\ V_T+UV_x+VV_Y+WV_z+P_Y &= 0,\\ W_T+UW_x+VW_Y+WW_z+\epsilon^{2/3}P_z &= 0,\\ U_x+V_Y+W_z &= 0.\end{aligned}\right\} \tag{3.1}$$

The form of the outer solution written in the inner variables suggests that the inner expansion is of the form

$$\left.\begin{aligned} U &= Y+\epsilon^{1/3}U_1+\epsilon^{2/3}U_2+\epsilon U_3+\dots,\\ V &= \epsilon^{1/3}V_1+\epsilon^{2/3}V_2+\epsilon V_3+\dots,\\ W &= \epsilon^{1/3}W_1+\epsilon^{2/3}W_2+\epsilon W_3+\dots,\\ P &= \epsilon^{-1/3}P_{-1}+P_0+\epsilon^{1/3}P_1+\epsilon^{2/3}P_2+\dots.\end{aligned}\right\} \tag{3.2}$$

We shall see that because of the scalings chosen in the outer expansion, the oblique waves will appear at an earlier order, namely at $O(\epsilon^{1/3})$ in the inner expansion compared with the plane wave, which first appears at $O(\epsilon^{2/3})$. Substituting this expansion into the governing equations, collecting powers of $\epsilon$ and grouping terms with the same $x$- and $z$-dependence, we arrive at a series of equations of the form

$$\phi_T+\mathrm{i}\frac{nY}{2}\phi = \chi(Y,T) \tag{3.3}$$

which have solutions of the form

$$\phi = \int_{-\infty}^{T} \chi(Y,T_0)\,\mathrm{e}^{n\mathrm{i}Y(T_0-T)/2}\,\mathrm{d}T_0. \tag{3.4}$$

We will also need to calculate the jumps in several quantities across the critical layer with, for example, the jump in $\phi$ given by $\int_{-\infty}^{\infty}\phi_Y\,\mathrm{d}Y$; when we come to do this, it should be recalled from the definition of the Fourier transform that, for real $a \neq 0$,

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}aYT}\,\mathrm{d}Y = \frac{2\pi}{|a|}\delta(T). \tag{3.5}$$

For the inner expansion, we shall use the notation $U_{l,m}^{(n)}$ to mean the term at $O(\epsilon^{n/3})$ multiplying $\exp[\mathrm{i}\alpha(lx+m\sqrt{3}z)/2]$; in what follows, we will give only the terms corresponding to non-negative $l$ and $m$, with the remaining terms following from symmetry: for example, $U_{-2,0}^{(2)} = U_{20}^{(2)*}$. Additionally, only those terms necessary to evaluate jumps will be determined.

### 3.1 $O(\epsilon^{1/3})$ terms

At this order, we find that the only terms present represent the oblique waves, with $V_{11}^{(1)} = A_{11}$ and

$$U_{11}^{(1)} = -\frac{3}{4}\int_{-\infty}^{T} A_{11}(T_0)\,\mathrm{e}^{\mathrm{i}Y(T_0-T)/2}\,\mathrm{d}T_0, \tag{3.6}$$

with $W_{11}^{(1)} = -U_{11}^{(1)}/\sqrt{3}$ and the relevant pressure terms given by $P_{11}^{(-1)} = iA_{11}/2$, $P_{11}^{(0)} = iC_{11}^{(2)\pm}/2$ and

$$P_{11}^{(1)} = -\tfrac{1}{4}iA_{11}Y^2 - A'_{11}Y + 6iA''_{11} - \tfrac{1}{2}i\alpha_1 C_{11}^{(2)\pm} + \tfrac{1}{2}iC_{11}^{(3)\pm} - 2D_{11}^{(2)\pm\prime}. \tag{3.7}$$

Since $P_{11}^{(1)}$ and $P_{11}^{(2)}$ must be continuous, this tells us that $C_{11}^{(2)+} = C_{11}^{(2)-} = C_{11}^{(2)}$ and $C_{11}^{(3)+} + 4iD_{11}^{(2)+\prime} = C_{11}^{(3)-} + 4iD_{11}^{(2)-\prime}$.

### 3.2. $O(\epsilon^{2/3})$ terms

At this order, it is necessary to calculate several terms. The two-dimensional wave is composed of both the original two-dimensional wave and the interaction of the oblique waves so that $V_{20}^{(2)}$ satisfies

$$V_{20,YT}^{(2)} + iYV_{20,Y}^{(2)} = -4U_{11}^{(1)}U_{11,Y}^{(1)} + 2iA_{11}U_{11,YY}^{(1)}, \tag{3.8}$$

which has a solution

$$V_{20}^{(2)} = \frac{3i}{2}\int_{-\infty}^{T}\int_{-\infty}^{T_1} \frac{-3T^2 + 3TT_0 - T_0^2 + 3TT_1 - T_0T_1 - T_1^2}{(T_0 + T_1 - 2T)^2} \times A_{11}(T_0)A_{11}(T_1)\,e^{iY(T_0+T_1-2T)/2}\,dT_0\,dT_1 + A_{20}(T). \tag{3.9}$$

The streamwise component $U_{20}^{(2)} = iV_{20,Y}^{(2)}$ and the pressure terms are given by $P_{20}^{(0)} = iA_{20}$, $P_{20}^{(1)} = iC_{20}^{(2)\pm}$ and

$$P_{20}^{(2)} = 6\int_{-\infty}^{T} \frac{A_{11}(T_0)A_{11}(T)\,e^{iY(T_0-T)/2}\,dT_0}{T_0 - T} + 3\int_{-\infty}^{T}\int_{-\infty}^{T_1} \frac{(T_0 - T_1)^2 A_{11}(T_0)A_{11}(T_1)\,e^{iY(T_0+T_1-2T)/2}\,dT_0\,dT_1}{(T_0 + T_1 - 2T)^4} - YA'_{20} + \tfrac{1}{2}i\alpha_1 Y^2 A_{20} + H_{20}^{(2)}(T). \tag{3.10}$$

Since $P_{20}^{(2)}$ must be continuous, this tells us that $C_{20}^{(2)+} = C_{20}^{(2)-} = C_{20}^{(2)}$. For the mean flow terms, we find that $V_{00}^{(2)} = W_{00}^{(2)} = 0$, $P_{00}^{(2)} = -4|A_{11}(T)|^2$, and

$$U_{00,T}^{(2)} = -2A_{11}^{*}U_{11,Y}^{(1)} - 2A_{11}U_{11,Y}^{(1)*}, \tag{3.11}$$

which has a solution

$$U_{00}^{(2)} = -\tfrac{1}{3}Y^3 + \frac{3i}{4}\int_{-\infty}^{T}\int_{-\infty}^{T} (T_0 - T_1)A_{11}^{*}(T_1)A_{11}(T_0)\,e^{iY(T_0-T_1)/2}\,dT_0\,dT_1. \tag{3.12}$$

For the oblique waves, it develops that $V_{11}^{(2)} = -\alpha_1 A_{11} + C_{11}^{(2)}$ and

$$U_{11T}^{(2)} + \tfrac{1}{2}iYU_{11}^{(2)} = -V_{11}^{(2)} - \tfrac{1}{2}i(P_{11}^{(0)} - \alpha_1 P_{11}^{(1)}) + \tfrac{1}{2}i\alpha_1 YU_{11}^{(1)}, \tag{3.13}$$

which has a solution

$$U_{11}^{(2)} = -\frac{3}{4}\int_{-\infty}^{T} ((T_0 - T)\alpha_1 A'_{11}(T_0) + C_{11}^{(2)}(T_0))\,e^{iY(T_0-T)/2}\,dT_0 \tag{3.14}$$

and also $W_{11}^{(2)} = -U_{11}^{(2)}/\sqrt{3}$, with the pressure term given by

$$P_{11}^{(2)} = iY^2(\tfrac{1}{2}\alpha_1 A_{11} - \tfrac{1}{4}C_{11}^{(2)}) + Y(\alpha_1 A'_{11} - C_{11}^{(2)\prime}) + H_{11}^{(2)}(T). \tag{3.15}$$

For the $\exp[\mathrm{i}\alpha(x+\sqrt{3}z)]$ terms, we find that $V_{22}^{(2)}=0, P_{22}^{(2)}=A_{11}^2$, and that

$$U_{22,T}^{(2)}+\mathrm{i}YU_{22}^{(2)}=-A_{11}U_{11,Y}^{(1)}, \tag{3.16}$$

which has a solution

$$U_{22}^{(2)}=\frac{3\mathrm{i}}{8}\int_{-\infty}^{T}\int_{-\infty}^{T_1}(T_0-T_1)A_{11}(T_1)A_{11}(T_0)\,\mathrm{e}^{\mathrm{i}Y(T_0+T_1-2T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1, \tag{3.17}$$

with $W_{22}^{(2)}=-U_{22}^{(2)}/\sqrt{3}$. There is also a cross-flow component, composed of $\exp(\mathrm{i}\sqrt{3}\,\alpha z)$ terms which generates the second term in (2.11) in the outer expansion, for which we find that

$$V_{02,YT}^{(2)}=2|U_{11}^{(1)}|^2-\mathrm{i}A_{11}^*U_{11,Y}^{(1)}+\mathrm{i}A_{11}U_{11,Y}^{(1)*}, \tag{3.18}$$

which has a solution

$$V_{02}^{(2)}=\frac{3\mathrm{i}}{4}\int_{-\infty}^{T}\int_{-\infty}^{T_1}\frac{2T_1+T_0-3T}{T_0-T_1}\times\left(A_{11}^*(T_1)A_{11}(T_0)\,\mathrm{e}^{\mathrm{i}Y(T_0-T_1)/2}+A_{11}^*(T_0)A_{11}(T_1)\,\mathrm{e}^{\mathrm{i}Y(-T_0+T_1)/2}\right)\mathrm{d}T_0\,\mathrm{d}T_1. \tag{3.19}$$

The quantity $W_{02}^{(2)}=\mathrm{i}V_{02,Y}^{(2)}/\sqrt{3}$ and

$$U_{02,T}^{(2)}=-V_{02}^{(2)}-A_{11}^*U_{11,Y}^{(1)}-A_{11}U_{11,Y}^{(1)*}, \tag{3.20}$$

which has a solution

$$U_{02}^{(2)}=\frac{3\mathrm{i}}{8}\int_{-\infty}^{T}\int_{-\infty}^{T_1}\frac{3T^2-2TT_0+T_0^2-4TT_1+2T_1^2}{T_0-T_1}\times\left(A_{11}^*(T_1)A_{11}(T_0)\,\mathrm{e}^{\mathrm{i}Y(T_0-T_1)/2}+A_{11}^*(T_0)A_{11}(T_1)\,\mathrm{e}^{\mathrm{i}Y(-T_0+T_1)/2}\right)\mathrm{d}T_0\,\mathrm{d}T_1 \tag{3.21}$$

with the pressure terms given by $P_{02}^{(1)}=C_{02}^{(1)\pm\prime}$ and

$$P_{02}^{(2)}=\frac{9}{2}\int_{-\infty}^{T}\int_{-\infty}^{T_1}\frac{\left(A_{11}^*(T_1)A_{11}(T_0)\,\mathrm{e}^{\mathrm{i}Y(T_0-T_1)/2}+A_{11}(T_1)A_{11}^*(T_0)\,\mathrm{e}^{\mathrm{i}Y(-T_0+T_1)/2}\right)}{(T_0-T_1)^2}\mathrm{d}T_0\,\mathrm{d}T_1+H_{02}^{(2)}(T). \tag{3.22}$$

Since $P_{02}^{(1)}$ must be continuous, this tells us that $C_{02}^{(1)+\prime}=C_{02}^{(1)-\prime}$. In the above, $H_{20}^{(2)}$, $H_{11}^{(2)}$, and $H_{02}^{(2)}$ are functions which do not need to be determined. We note that there is a non-zero jump in $V_{02}^{(2)}$ across the critical layer,

$$\int_{-\infty}^{\infty}V_{02,Y}^{(2)}\,\mathrm{d}Y=\frac{9\pi}{2}\int_{-\infty}^{T}\int_{-\infty}^{T_1}|A_{11}(T_0)|^2\,\mathrm{d}T_0\,\mathrm{d}T_1, \tag{3.23}$$

and it is this jump which necessitates the presence in the outer expansion of the cross-stream terms such as the second term in (2.11).

### 3.3. $O(\epsilon)$ *terms*

It is at this order that we shall evaluate the jump in the oblique waves. From the outer expansion, we know that

$$A_{11}=\frac{1}{2\alpha_1}(D_{11}^{(2)+}-D_{11}^{(2)-})=\frac{1}{8\mathrm{i}\alpha_1}\int_{-\infty}^{\infty}(U_{11,Y}^{(3)}+\sqrt{3}\,W_{11,Y}^{(3)}+2\mathrm{i}A_{11})\,\mathrm{d}Y. \tag{3.24}$$

We find that

$$\left(\frac{\partial}{\partial T}+\frac{\mathrm{i}Y}{2}\right)(U^{(3)}_{11,Y}+\sqrt{3}\,W^{(3)}_{11,Y}) = \mathscr{F}^{(3)}_{11}, \tag{3.25}$$

where the forcing term $\mathscr{F}^{(3)}_{11}$ is given in the Appendix, so that

$$\begin{aligned}
&U^{(3)}_{11,Y}+\sqrt{3}\,W^{(3)}_{11,Y}+2\mathrm{i}A_{11}\\
&\quad = 2U^{(2)}_{02}\,U^{(1)}_{11,YY}+2U^{(1)*}_{11,YY}\,U^{(2)}_{20}\\
&\qquad +\frac{3\mathrm{i}}{64}\int_{-\infty}^{T}\int_{-\infty}^{T_2}\int_{-\infty}^{T_1} K^{(3a)}_{11}(T_0,T_1,T_2)\,A_{11}(T_2)\,A_{11}(T_1)\,A^{*}_{11}(T_0)\,\mathrm{e}^{\mathrm{i}Y(-T_0+T_1+T_2-T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1\,\mathrm{d}T_2\\
&\qquad +\frac{3\mathrm{i}}{64}\int_{-\infty}^{T}\int_{-\infty}^{T_2}\int_{-\infty}^{T_1} K^{(3b)}_{11}(T_0,T_1,T_2)\,A_{11}(T_2)\,A^{*}_{11}(T_1)\,A_{1}(T_0)\,\mathrm{e}^{\mathrm{i}Y(T_0-T_1+T_2-T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1\,\mathrm{d}T_2\\
&\qquad +\frac{3\mathrm{i}}{64}\int_{-\infty}^{T}\int_{-\infty}^{T_2}\int_{-\infty}^{T_1} K^{(3c)}_{11}(T_0,T_1,T_2)\,A^{*}_{11}(T_2)\,A_{11}(T_1)\,A_{11}(T_0)\,\mathrm{e}^{\mathrm{i}Y(T_0+T_1-T_2-T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1\,\mathrm{d}T_2\\
&\qquad +4\mathrm{i}\int_{-\infty}^{T} A'_{11}(T_0)\,\mathrm{e}^{\mathrm{i}Y(T_0-T)/2}\,\mathrm{d}T_0\\
&\qquad -\frac{3}{8}\int_{-\infty}^{T}\int_{-\infty}^{T_1}(T_0-T_1)^2A_{20}(T_1)\,A^{*}_{11}(T_0)\,\mathrm{e}^{\mathrm{i}Y(-T_0+2T_1-T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1,
\end{aligned} \tag{3.26}$$

and the kernels $K^{(3)}_{11}$ are given in the Appendix. The term $2U^{(2)}_{02}\,U^{(1)}_{11,YY}+2U^{(1)*}_{11,YY}\,U^{(2)}_{20}$ is separated from the remainder of the expression because this greatly simplifies the evaluation of the jump in $(U^{(3)}_{11}+\sqrt{3}\,W^{(3)}_{11}+2\mathrm{i}A_{11}\,Y)$ across the critical layer. This jump is

$$\begin{aligned}
&\int_{-\infty}^{\infty}(U^{(3)}_{11,Y}+\sqrt{3}\,W^{(3)}_{11,Y}+2\mathrm{i}A_{11})\,\mathrm{d}Y\\
&\quad = -\frac{3\mathrm{i}\pi}{16}\int_{0}^{\infty}\int_{0}^{\infty}\tau_0(4\tau_0^2+5\tau_0\tau_1+3\tau_1^2)\,A_{11}(T-\tau_0)\,A^{*}_{11}(T-2\tau_0-\tau_1)\,A_{11}(T-\tau_0-\tau_1)\,\mathrm{d}\tau_0\,\mathrm{d}\tau_1\\
&\qquad -\frac{3\pi}{2}\int_{0}^{\infty}\tau_0^2A_{20}(T-\tau_0)\,A^{*}_{11}(T-2\tau_0)\,\mathrm{d}\tau_0+8\mathrm{i}\pi A'_{11},
\end{aligned} \tag{3.27}$$

and hence the amplitude equation for the oblique waves is

$$\begin{aligned}
&\frac{\mathrm{d}A_{11}}{\mathrm{d}T}-\frac{\alpha_1}{\pi}A_1\\
&\quad = \frac{3}{128}\int_{0}^{\infty}\int_{0}^{\infty}\tau_0(4\tau_0^2+5\tau_0\tau_1+3\tau_1^2)\,A_{11}(T-\tau_0)\,A^{*}_{11}(T-2\tau_0-\tau_1)\,A_{11}(T-\tau_0-\tau_1)\,\mathrm{d}\tau_0\,\mathrm{d}\tau_1\\
&\qquad -\frac{3\mathrm{i}}{16}\int_{0}^{\infty}\tau_0^2A_{20}(T-\tau_0)\,A^{*}_{11}(T-2\tau_0)\,\mathrm{d}\tau_0.
\end{aligned} \tag{3.28}$$

It is also necessary to calculate $U^{(3)}_{11}$ and $W^{(3)}_{11}$. From the $z$-momentum equation, we have

$$W^{(3)}_{11,T}+\tfrac{1}{2}\mathrm{i}Y\,W^{(3)}_{11} = \mathscr{G}^{(3)}_{11} \tag{3.29}$$

which has a solution

$$W_{11}^{(3)} = 3^{-1/2}U_{11,Y}^{(1)}\,U_{02}^{(2)}+3^{-1/2}U_{11,Y}^{(1)*}\,U_{20}^{(2)}+\frac{\mathrm{i}Y}{4\sqrt{3}}A_{11}(T)+\frac{7}{2\sqrt{3}}A_{11}'(T)$$
$$+\frac{\alpha_1\sqrt{3}}{4}\int_{-\infty}^{T}(C_{11}^{(2)}(T_0)+(T_0-T)\,C_{11}^{(2)\prime}(T_0))\,\mathrm{e}^{\mathrm{i}Y(T_0-T)/2}\,\mathrm{d}T_0$$
$$+\frac{\sqrt{3}}{4}\int_{-\infty}^{T}(C_{11}^{(3)\pm}(T_0)+4\mathrm{i}D_{11}^{(2)\pm\prime}(T_0))\,\mathrm{e}^{\mathrm{i}Y(T_0-T)/2}\,\mathrm{d}T_0$$
$$-\frac{\sqrt{3}\,\alpha_1^2}{8}\int_{-\infty}^{T}(2(T_0-T)\,A_{11}'(T_0)+(T_0-T)^2A_{11}''(T_0))\,\mathrm{e}^{\mathrm{i}Y(T_0-T)/2}\,\mathrm{d}T_0$$
$$+\frac{1}{2\sqrt{3}}\int_{-\infty}^{T}(9A_{11}''(T_0)-2(T_0-T)\,A_{11}'''(T_0))\,\mathrm{e}^{\mathrm{i}Y(T_0-T)/2}\,\mathrm{d}T_0$$
$$-\frac{\sqrt{3}\mathrm{i}}{8}\int_{-\infty}^{T}\int_{-\infty}^{T_1}(T_0-T_1)\,A_{20}(T_1)\,A_{11}^{*}(T_0)\,\mathrm{e}^{\mathrm{i}Y(-T_0+2T_1-T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1$$
$$+\frac{\sqrt{3}}{64}\int_{-\infty}^{T}\int_{-\infty}^{T_2}\int_{-\infty}^{T_1}L_{11}^{(3a)}A_{11}(T_2)\,A_{11}(T_1)\,A_{11}^{*}(T_0)\,\mathrm{e}^{\mathrm{i}Y(-T_0+T_1+T_2-T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1\,\mathrm{d}T_2$$
$$+\frac{\sqrt{3}}{64}\int_{-\infty}^{T}\int_{-\infty}^{T_2}\int_{-\infty}^{T_1}L_{11}^{(3b)}A_{11}(T_2)\,A_{11}^{*}(T_1)\,A_{11}(T_0)\,\mathrm{e}^{\mathrm{i}Y(T_0-T_1+T_2-T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1\,\mathrm{d}T_2$$
$$+\frac{\sqrt{3}}{64}\int_{-\infty}^{T}\int_{-\infty}^{T_2}\int_{-\infty}^{T_1}L_{11}^{(3c)}A_{11}^{*}(T_2)\,A_{11}(T_1)\,A_{11}(T_0)\,\mathrm{e}^{\mathrm{i}Y(T_0+T_1-T_2-T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1\,\mathrm{d}T_2, \quad (3.30)$$

where the kernels $L_{11}^{(3)}$ are given in the Appendix. We can also calculate $U_{11,Y}^{(3)}$ from the above and, in addition, we must calculate some of the $\exp[\mathrm{i}\alpha(3x/2+\sqrt{3}z/2)]$ terms. From the $z$-momentum equation, we have

$$W_{31,T}^{(3)}+\tfrac{3}{2}\mathrm{i}Y\,W_{31}^{(3)} = \mathrm{i}\sqrt{3}\,W_{11}^{(1)}\,W_{22}^{(2)}-A_{11}\,W_{22,Y}^{(2)}-\tfrac{1}{2}\mathrm{i}U_{20}^{(2)}\,W_{11}^{(1)}-V_{20}^{(2)}\,W_{11,Y}^{(1)}, \quad (3.31)$$

which has a solution

$$W_{31}^{(3)} = U_{20}^{(2)}\,W_{11,Y}^{(1)}$$
$$-\frac{\sqrt{3}\mathrm{i}}{8}\int_{-\infty}^{T}\int_{-\infty}^{T_1}(T_0-T_1)\,A_{20}(T_1)\,A_{11}(T_0)\,\mathrm{e}^{\mathrm{i}Y(T_0+2T_1-3T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1$$
$$+\frac{\sqrt{3}}{64}\int_{-\infty}^{T}\int_{-\infty}^{T_2}\int_{-\infty}^{T_1}K_{31}^{(3)}\,A_{11}(T_2)\,A_{11}(T_1)\,A_{11}(T_0)\,\mathrm{e}^{\mathrm{i}Y(T_0+T_1+T_2-3T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1\,\mathrm{d}T_2, \quad (3.32)$$

with the kernel given in the Appendix. We also find that

$$U_{31,YT}^{(3)}+\tfrac{3}{2}\mathrm{i}\,YU_{31,Y}^{(3)} = \mathscr{G}_{31}^{(3)}, \quad (3.33)$$

which has a solution

$$U_{31,Y}^{(3)} = -\frac{3}{16}\int_{-\infty}^{T}\int_{-\infty}^{T_1}(T_0-T)(T_0-T_1)\,A_{20}(T_1)\,A_{11}(T_0)\,\mathrm{e}^{\mathrm{i}Y(T_0+2T_1-3T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1$$
$$+\frac{3\mathrm{i}}{128}\int_{-\infty}^{T}\int_{-\infty}^{T_2}\int_{-\infty}^{T_1}L_{31}^{(3a)}\,A_{11}(T_2)\,A_{11}(T_1)\,A_{11}(T_0)\,\mathrm{e}^{\mathrm{i}Y(T_0+T_1+T_2-3T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1\,\mathrm{d}T_2$$
$$+\frac{3\mathrm{i}}{128}\int_{-\infty}^{T}\int_{-\infty}^{T_2}\int_{-\infty}^{T_1}L_{31}^{(3b)}\,A_{11}(T_2)\,A_{11}(T_1)\,A_{11}(T_0)\,\mathrm{e}^{\mathrm{i}Y(T_0+T_1+T_2-3T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1\,\mathrm{d}T_2$$
$$+U_{20}^{(2)}\,U_{11,YY}^{(1)}+\tfrac{1}{3}U_{20,Y}^{(2)}\,U_{11,Y}^{(1)}. \quad (3.34)$$

The kernels in (3.34) are given in the Appendix.

For the $\exp(\mathrm{i}\alpha x)$ terms, we need only calculate $U_{20}^{(3)}$, which satisfies

$$U_{20,YT}^{(3)}+\mathrm{i}U_{20,Y}^{(3)} = \mathrm{i}\alpha_1 YU_{20,Y}^{(2)}+4\mathrm{i}\alpha_1 U_{11}^{(1)}U_{11,Y}^{(1)}$$
$$-2A_{11}U_{11,YY}^{(2)}+2\alpha_1 A_{11}U_{11,YY}^{(1)}-2C_{11}^{(2)}U_{11,YY}^{(1)}-4\mathrm{i}\frac{\partial}{\partial Y}(U_{11}^{(2)}U_{11}^{(1)}), \quad (3.35)$$

which has a solution

$$U_{20}^{(3)} = \frac{3\alpha_1\mathrm{i}}{4}\int_{-\infty}^{T}\int_{-\infty}^{T_1}\frac{(T_0-T_1)(5T_0-4T_1-T)}{T_0+T_1-2T}A_{11}(T_1)A_{11}(T_0)\,\mathrm{e}^{\mathrm{i}Y(T_0+T_1-2T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1$$
$$+\frac{3\alpha_1\mathrm{i}}{8}\int_{-\infty}^{T}\int_{-\infty}^{T_1}K_{20}^{(3)}A_{11}(T_1)A_{11}'(T_0)\,\mathrm{e}^{\mathrm{i}Y(T_0+T_1-2T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1$$
$$-\frac{3\mathrm{i}}{4}\int_{-\infty}^{T}\int_{-\infty}^{T}\frac{-3T^2+3TT_0-T_0^2+3TT_1-T_0T_1-T_1^2}{T_0+T_1-2T}$$
$$\times C_{11}^{(2)}(T_1)A_{11}(T_0)\,\mathrm{e}^{\mathrm{i}Y(T_0+T_1-2T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1$$
$$+\frac{3\mathrm{i}\alpha_1}{8}\int_{-\infty}^{T}\int_{-\infty}^{T_1}\frac{(T_1-T)^2(3T_0+T_1-4T)}{T_0+T_1-2T}A_{11}'(T_1)A_{11}(T_0)\,\mathrm{e}^{\mathrm{i}Y(T_0+T_1-2T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1. \quad (3.36)$$

### 3.4. $O(\epsilon^{4/3})$ *terms*

It is at this order that we shall evaluate the jump in the plane wave. From the outer expansion, we know that

$$A_{20} = \frac{1}{2\alpha_1}(D_{20}^{(2)+}-D_{20}^{(2)-}) = \frac{1}{4\mathrm{i}\alpha_1}\int_{-\infty}^{\infty}(U_{20,Y}^{(4)}+\mathrm{i}A_{20})\,\mathrm{d}Y. \quad (3.37)$$

We find that $U_{20}^{(4)}$ satisfies

$$U_{20,YT}^{(4)}+\mathrm{i}YU_{20,Y}^{(4)}+\mathrm{i}A_{20}'-A_{20}Y = \mathscr{F}_{20}^{(4)}, \quad (3.38)$$

where $\mathscr{F}_{20}^{(4)}$ is given in the Appendix and after extensive integration by parts, we find that (3.38) has a solution which can be written in the form

$$U_{20}^{(4)}+\mathrm{i}A_{20}Y = \tilde{U}_{20}^{(4)}+U_{02}^{(2)}\left(2U_{22,YY}^{(2)}-\frac{\partial^3}{\partial Y^3}(U_{11}^{(1)2})\right)$$
$$+U_{20}^{(2)}\left(U_{00,YY}^{(2)}-2\frac{\partial^3}{\partial Y^3}(U_{11}^{(1)}U_{11}^{(1)*})\right)+2\mathrm{i}\int_{-\infty}^{T}A_{20}'(T_0)\,\mathrm{e}^{\mathrm{i}Y(T_0-T)}\,\mathrm{d}T_0$$
$$+\frac{3\mathrm{i}}{64}\int_{-\infty}^{T}\int_{-\infty}^{T_2}\int_{-\infty}^{T_1}K_{20}^{(4a)}A_{11}(T_2)A_{20}(T_1)A_{11}^*(T_0)$$
$$\times\mathrm{e}^{\mathrm{i}Y(-T_0+2T_1+T_2-2T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1\,\mathrm{d}T_2$$
$$-\frac{3\mathrm{i}}{64}\int_{-\infty}^{T}\int_{-\infty}^{T_2}\int_{-\infty}^{T_1}(T_0-T_1)^3A_{20}(T_2)A_{11}(T_1)A_{11}^*(T_0)$$
$$\times\mathrm{e}^{\mathrm{i}Y(-T_0+T_1+2T_2-2T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1\,\mathrm{d}T_2$$
$$+\frac{9}{256}\int_{-\infty}^{T}\int_{-\infty}^{T_3}\int_{-\infty}^{T_2}\int_{-\infty}^{T_1}K_{20}^{(4b)}(T_0,T_1,T_2,T_3)$$
$$\times A_{11}(T_3)A_{11}(T_2)A_{11}(T_1)A_{11}^*(T_0)\,\mathrm{e}^{\mathrm{i}Y(-T_0+T_1+T_2+T_3-2T)/2}\,\mathrm{d}T_0\,\mathrm{d}T_1\,\mathrm{d}T_2\,\mathrm{d}T_3, \quad (3.39)$$

where the kernels are given in the Appendix.

The quantity $\tilde{U}_{20}^{(4)}$ consists of terms which do not contribute to the jump across the critical layer, and the remaining terms on the first line of (3.39) are separated from the remainder of the expression because, as was the case with the $O(\epsilon)$ terms, this greatly simplifies the evaluation of the jump in $(U_{20}^{(4)}+\mathrm{i}A_{20}\,Y)$ across the critical layer. This jump is

$$\int_{-\infty}^{\infty}(U_{20,Y}^{(4)}+\mathrm{i}A_{20})\,\mathrm{d}Y=4\mathrm{i}\pi A'_{2,0}$$
$$+\frac{3\mathrm{i}\pi}{2}\int_0^\infty\int_0^\infty \tau_0^3 A_{20}(T-\tau_0)A_{11}^*(T-3\tau_0-\tau_1)A_{11}(T-\tau_0-\tau_1)\,\mathrm{d}\tau_0\,\mathrm{d}\tau_1$$
$$+\frac{3\mathrm{i}\pi}{4}\int_0^\infty\int_0^\infty \tau_0(\tau_0+\tau_1)(2\tau_0+\tau_1)A_{11}(T-\tau_0)$$
$$\times A_{20}(T-\tau_0-\tau_1)A_{11}^*(T-3\tau_0-2\tau_1)\,\mathrm{d}\tau_0\,\mathrm{d}\tau_1$$
$$+\frac{3\pi}{32}\int_0^\infty\int_0^\infty\int_0^\infty (3\tau_2^4+(4\tau_0+8\tau_1)\tau_2^3+(3\tau_0^2+\tau_0\tau_1+2\tau_1^2)$$
$$\times\tau_2^2-(3\tau_0^2\tau_1+5\tau_0\tau_1^2+4\tau_1^3)\tau_2)$$
$$\times A_{11}(T-\tau_2)A_{11}(T-\tau_2-\tau_1)A_{11}(T-\tau_2-\tau_1-\tau_0)$$
$$\times A_{11}^*(T-3\tau_2-2\tau_1-\tau_0)\,\mathrm{d}\tau_0\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2, \tag{3.40}$$

and hence the equation for the plane wave is

$$\frac{\mathrm{d}A_{20}}{\mathrm{d}T}-\frac{\alpha_1}{\pi}A_{20}=\frac{3}{8}\int_0^\infty\int_0^\infty \tau_0^3 A_{20}(T-\tau_0)A_{11}^*(T-3\tau_0-\tau_1)A_{11}(T-\tau_0-\tau_1)\,\mathrm{d}\tau_0\,\mathrm{d}\tau_1$$
$$+\frac{3}{16}\int_0^\infty\int_0^\infty \tau_0(\tau_0+\tau_1)(2\tau_0+\tau_1)A_{11}(T-\tau_0)$$
$$\times A_{20}(T-\tau_0-\tau_1)A_{11}^*(T-3\tau_0-2\tau_1)\,\mathrm{d}\tau_0\,\mathrm{d}\tau_1$$
$$-\frac{3\mathrm{i}}{128}\int_0^\infty\int_0^\infty\int_0^\infty [3\tau_2^4+(4\tau_0+8\tau_1)\tau_2^3$$
$$+(3\tau_0^2+\tau_0\tau_1+2\tau_1^2)\tau_2^2-(3\tau_0^2\tau_1+5\tau_0\tau_1^2+4\tau_1^3)\tau_2]$$
$$\times A_{11}(T-\tau_2)A_{11}(T-\tau_2-\tau_1)A_{11}(T-\tau_2-\tau_1-\tau_0)$$
$$\times A_{11}^*(T-3\tau_2-2\tau_1-\tau_0)\,\mathrm{d}\tau_0\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2. \tag{3.41}$$

Thus, we have two coupled nonlinear integro-differential equations (3.28) and (3.41) governing the evolution of $A_{20}$ and $A_{11}$. It should be noted that for the tanh $y$ mixing layer studied here, the constants multiplying the quadratic term in (3.28) and the quartic term in (3.41) are imaginary and the constants multiplying the cubic term in (3.28) and the cubic terms in (3.41) are real. By contrast, for the adverse-pressure-gradient boundary layer (Goldstein & Lee 1992), the corresponding coefficients are all imaginary, which Goldstein & Lee attribute to the fact that they studied the long-wave limit, and for the Stokes layer (Wu 1992) they are all complex. Numerical results obtained from the solution of the amplitude equations for various values of the relevant parameters are discussed in considerable detail in §5 of Goldstein & Lee. Somewhat surprisingly, the qualitative behaviour of the solutions does not depend on the coefficients being real, as shown by the additional computations of Wu.

As previously observed by Goldstein & Lee (1992) and Wu (1992), the solutions to these two equations develop a singularity and blow up at some finite time, say $T_s$. It was shown that near to this singularity, the solutions have the asymptotic forms

$$A_{20} \sim b_{20}/(T_s - T)^{4+2i\psi} \quad \text{and} \quad A_{11} \sim b_{11}/(T_s - T)^{3+i\psi}, \tag{3.42}$$

where $\psi$ is a real constant and $b_{20}$ and $b_{11}$ are complex; $\psi$, $|b_{20}|$, $|b_{22}|$ and a relationship between the arguments of $b_{20}$ and $b_{11}$ can be obtained by substituting this asymptotic form into (3.28) and (3.41). Although Wu links this singularity to the appearance of turbulent bursts in experiments on Stokes layers, it seems most likely that its significance in our case is, rather, a breakdown in the theory signalling a more nonlinear stage governed by the Euler equations, as suggested by Goldstein & Lee. The singularity may also be associated in the case of the mixing layer with the rapid thickening observed to occur near the sites of vortex pairing. The addition of viscous effects to the theory would no doubt modify this behaviour, but the present result is still meaningful at moderate Reynolds numbers according to the investigation by Wu, Lee & Cowley (1993) of the case where only the oblique waves are present.

## 4. Concluding remarks

In the preceding sections, coupled amplitude evolution equations were derived describing the interaction of a triad of modes comprising a plane fundamental and a pair of oblique subharmonics that in the linear neutral limit satisfy exactly the conditions for triad resonance. Principal differences between our approach and previous analyses of plane mixing layers reviewed in §1 are that (i) we do not regard the subharmonic as a small perturbation of a spatially periodic flow, i.e. it is not treated as a secondary instability as in Kelly (1967) or Pierrehumbert & Widnall (1982); and (ii) we do not restrict the shorter (fundamental) wave to be neutral. In fact, it is essential that all modes be slightly non-neutral on a linear basis because of our utilization of a non-equilibrium critical layer.

To appreciate the significance of the latter point, note that it is only with such a critical layer that the coefficient of the back-reaction term vanishes in (1.2), the plane wave evolution equation for $t < O(\epsilon^{1/3})$. This means that the parametric resonance observed in the experiments of Corke & Kusek (1993) can occur even when the amplitude of the oblique waves grows to become comparable with that of the plane wave. By contrast, the approximation to the usual triad equations discussed by these authors is valid only so long as $A_{11} \ll A_{20}$. It is therefore now clear that in order to predict some important aspects of free-shear-layer transition (or to relate observation with theory) it is necessary to specify the critical-layer régime (i.e. viscous, nonlinear, wave packet, non-equilibrium or some combination thereof). On the other hand, the analyses of Monkewitz (1988) and those cited in the preceding paragraph completely neglect critical-layer effects, yet still are capable of describing the onset of certain phenomena. This is partly because critical layers are not significant while instabilities are evolving rapidly; they become important subsequently owing to nonlinear saturation and to the spreading of the shear layer which causes the wave number of the initially most amplified disturbance to tend, on a non-dimensional basis, toward the neutral value.

Our own analysis, of course, also has several limitations. Being inviscid, it assumes relatively large Reynolds numbers and, in addition, the initial amplitudes of the oblique waves are assumed to be of the same order of magnitude as the plane wave. When these conditions are satisfied, the amplitude equations (3.28) and (3.41) describe three successive stages of the transition process. These are linear instability, parametric resonance for $t \sim O(\epsilon^{-1/4})$ and, finally, a fully coupled stage when $t \sim O(\epsilon^{-1/3})$. In the spatial case discussed below, the same scalings apply with $t$ replaced by $x$. During the fully coupled stage, our analysis produces streamwise vortices. Hence, the experimental observation of such vortices can be explained by resonant interaction theory as an alternative to the translative instability mechanism proposed by Pierrehumbert & Widnall (1982).

Clearly, our results are not in accord with the scenario described in the opening paragraph of §1 in which three-dimensional effects do not arise until after the plane modes produced by the inflexional instability have reached equilibrium. (We deliberately avoid using the term 'Kelvin–Helmholtz instability' which properly refers to a quite different mechanism describing the instability of a vortex sheet between two streams of different density.) Experimentalists, as discussed in §1, have already shown that the forcing of oblique waves can radically alter this sequence of events. Often, the initial conditions in these experiments were not close enough to those in our theoretical model to permit detailed comparisons. However, the most recent experiments reported by Nygaard & Glezer (1994) involve pairs of oblique waves inclined to the mean flow that are of comparable amplitude to the plane wave and so it should be possible to relate their observations to our model. The schlieren photographs in figure 17, for example, correspond to a disturbance configuration similar to our own, but the obliqueness angle is much smaller than 60°. Although the qualitative behaviour in these experimental observations seems to agree in several respects with our theory, there are, unfortunately, no data presented for growth rates.

At several points in Nygaard & Glezer (1994), the apparent existence of a cutoff spanwise wavenumber for forced oblique waves is noted. Usually this corresponds to an angle of about 45° which is said to agree reasonably well with the result for the helical pairing instability investigated by Pierrehumbert & Widnall. The actual value depends on a parameter denoted $\rho$ in the Stuart vortex model and the result for $\rho = 0.25$, for example, is equivalent to an angle of 65°. This is quite close to the 60° value which yields exact triad resonance in our own analysis. An important difference, however, is that the maximum amplification rate in their model occurs for a plane-wave subharmonic, whereas our results favour oblique waves. It would be of interest to calculate the obliqueness angle at which the growth rate predicted by (3.28), modified for detuning, is a maximum for a particular set of experimental data. The result is not obvious because of competing effects; as the obliqueness angle is decreased, the linear amplification rises, but the resonance effect is diminished by detuning.

Also of interest are the direct numerical simulations of Shoppe *et al.* (1994) because, as noted in our previous article (Mallier & Maslowe 1994), numerical studies reported up to the time when that article was written viewed the oblique modes as secondary instabilities and, therefore, the latter were not of sufficient amplitude to permit comparison with the present analysis. Shoppe *et al.*, however, do have some results for a case with a disturbance configuration similar to that considered herein, with the oblique waves inclined at $\pm 45°$ to the mean flow. They have examined the effect of increasing the energy in the oblique waves, comparing small values with a case where energy in the oblique waves is 50% of that in the plane-wave fundamental mode. A very rapid amplification of the oblique waves was observed, as well as a substantial thickening of the shear layer. Although our analysis can easily be modified to account for the inclination of the oblique waves being 45° instead of 60°, we do not incorporate a variable shear-layer thickness. Perhaps this feature can be implemented in a future model. In any case, the agreement with regard to amplification of the oblique waves is encouraging.

Returning now to the case where three-dimensional effects do not become significant until after the pairing of spanwise vortices has occurred, a resonant triad analysis may still be appropriate. If $\epsilon$ is an amplitude parameter characteristic of the plane wave and $\delta$ is the amplitude of the oblique waves, then the condition $\delta \ll \epsilon^3$ ensures that the plane wave saturates before the onset of a stage where the modes are fully coupled. The evolution of a plane wave from the exponential amplification to nonlinear saturation can be described accurately using a non-equilibrium nonlinear critical layer as was shown convincingly by Hultgren (1992). Outside the critical layer, his analysis is linear and includes weak non-parallelism. Hultgren's solution could be expanded to incorporate a pair of oblique waves which grow parametrically in the sense that the plane wave is unaffected by their presence until after equilibrium. An analysis of the adverse-pressure-gradient boundary layer along these lines was given recently by Wundrow, Hultgren & Goldstein (1994) and a similar development for free shear layers would probably best describe experiments with two-dimensional forcing.

A description based on spatial rather than temporal evolution is, of course, usually most appropriate for comparison with experiments on free shear layers. As discussed in the appendix of Kelly (1967), the constant $u_m$ in the velocity profile (2.1) must be greater than zero in order to have spatial instability. Only minor modifications are required to adapt our amplitude equations to the spatial case. In a frame of reference moving with the real phase speed (assuming it is the same for both the plane and oblique waves), the form of the evolution equations would be unchanged. The independent variable would be $X = \epsilon^{1/3}x$ instead of $T$ and, in addition, the constants appearing in front of the nonlinear terms, as well as those multiplying $A_{20}$ and $A_{11}$, would now be complex. This can be seen easily by recalling that the evolution equation for a wave packet is of the form

$$\frac{\partial A}{\partial T}+c_g\frac{\partial A}{\partial X} = \gamma A + \text{nonlinear terms}, \tag{5.1}$$

where $c_g$ is the group velocity (see e.g. Benney & Maslowe 1975 or Huerre 1987). Using the plane wave as an illustration, it is known that $c_g = (1-2\mathrm{i}/\pi)$ when $u_m = 1$. Consequently, converting (3.41) to an equation for $\mathrm{d}A_{20}/\mathrm{d}X$ involves multiplication of all other terms by the complex constant $c_g^{-1}$. As noted near the end of §3, this does not change the qualitative behaviour of solutions to the amplitude equations.

Let us now discuss briefly the question raised in §1 of dispersive effects in the spatial case. Whereas $c_r = u_m$ for all unstable as well as neutral modes in the temporal problem, numerical solutions of Rayleigh's equation show that $c_r$ varies with $\alpha_r$ when there is spatial growth and that $|c_r - u_m|$ is greatest when $\alpha_r \to 0$. However, it was observed earlier in this section that the growth of the shear-layer thickness displaces the effective dimensionless wavenumber (and frequency) toward larger values where dispersion is less pronounced. This is probably why temporal numerical simulations appear to provide a satisfactory description of experimental events despite some important differences in the linear theory of spatial and temporal modes.

Finally, we consider briefly the effects of a departure of the inclination of the oblique waves from the 60° angle at which exact resonance occurs. Again writing $\alpha = 1-\epsilon^{1/3}\alpha_1$, as in (1.1), but now expanding the spanwise wavenumber and phase speed as

$$\beta = \tfrac{1}{2}\surd 3(1-\epsilon^{1/3}\beta_1) \quad \text{and} \quad c = 1-\epsilon^{1/3}c_1 \tag{5.2}$$

allows us to treat the off-resonance case with spatial evolution. The crucial point is that for the interaction to be most effective the system of waves should propagate downstream with all components moving at the same speed. This ensures, also, that they share a common critical layer. It turns out that the condition for this to be so is that

$$\beta_1 = 5\alpha_1, \tag{5.3}$$

in which case

$$c_1 = 4\alpha_1/\pi^2. \tag{5.4}$$

Even when $1-\alpha$ is not particularly small, numerical solutions of Rayleigh's equation reveal that an obliqueness angle can generally be found such that all components of the triad have the same phase speed and the resonance mechanism will, as a result, still be active.

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## Appendix

The forcing terms for §3 are given by

$$\begin{aligned}
\mathscr{F}^{(3)}_{11} = {} & 2\mathrm{i}A'_{11}(T) - YA_{11}(T) \\
& + \tfrac{1}{2}\mathrm{i}\alpha_1\, Y(U^{(2)}_{11,Y} + \surd 3\, W^{(2)}_{11,Y}) - 2(\partial/\partial Y)(\mathrm{i}U^{(1)}_{11}\, U^{(2)}_{02} + U^{(1)}_{11,Y}\, V^{(2)}_{02} + U^{(1)*}_{11,Y}\, V^{(2)}_{20}) \\
& - (\partial^2/\partial Y^2)(A_{11}(T)(U^{(2)}_{02} + \surd 3\, W^{(2)}_{02} + U^{(2)}_{00}) + A^*_{11}(T)\, U^{(2)}_{20}), \\
\mathscr{G}^{(3)}_{11} = {} & -A^*_{11}\, W^{(2)}_{22,Y} - \tfrac{1}{2}\mathrm{i}U^{(2)}_{20}\, W^{(1)*}_{11} + V^{(2)}_{20}\, W^{(1)*}_{11,Y} - \tfrac{1}{2}\mathrm{i}U^{(2)}_{00}\, W^{(1)}_{11} \\
& - A_{11}\, W^{(2)}_{02,Y} + \tfrac{1}{2}\mathrm{i}U^{(2)}_{02}\, W^{(1)}_{11} + V^{(2)}_{02}\, W^{(1)}_{11,Y} + \tfrac{1}{2}\mathrm{i}\surd 3\, W^{(2)}_{02}\, W^{(1)}_{11} \\
& + \tfrac{1}{2}\mathrm{i}\alpha_1\, YW^{(2)}_{11} - \tfrac{1}{2}\mathrm{i}\,\surd 3\, P^{(1)}_{11} + \tfrac{1}{2}\mathrm{i}\,\surd 3\,\alpha_1\, P^{(0)}_{11}, \\
\mathscr{G}^{(3)}_{31} = {} & -(1/\surd 3)(W^{(3)}_{31,YT} + \tfrac{3}{2}\mathrm{i}\, YW^{(3)}_{31,Y}) - \tfrac{8}{3}\mathrm{i}U^{(1)}_{11,Y}\, U^{(2)}_{22} - \tfrac{8}{3}\mathrm{i}U^{(1)}_{11}\, U^{(2)}_{22,Y} - \tfrac{2}{3}A_{11}\, U^{(2)}_{22,YY} \\
& - \tfrac{4}{3}\mathrm{i}U^{(2)}_{20,Y}\, U^{(1)}_{11} + \tfrac{2}{3}V^{(2)}_{20,Y}\, U^{(1)}_{11,Y} - \tfrac{2}{3}V^{(2)}_{20}\, U^{(1)}_{11,YY} - A_{11}\, U^{(2)}_{20,YY}, \\
\mathscr{F}^{(4)}_{20} = {} & \mathrm{i}\alpha_1\, YU^{(3)}_{20,Y} - \mathrm{i}P^{(2)}_{20,Y} + \mathrm{i}A'_{20} - A_{20}\, Y + 4\mathrm{i}\alpha_1(\partial/\partial Y)(U^{(1)}_{11}\, U^{(2)}_{11}) \\
& - 4\mathrm{i}(\partial/\partial Y)(U^{(1)}_{11}\, U^{(3)}_{11}) - 2A_{11}\, U^{(3)}_{11,YY} - 2(\partial^2/\partial Y^2)(V^{(3)}_{11}\, U^{(1)}_{11}) \\
& - 4\mathrm{i}(\partial/\partial Y)(U^{(1)*}_{11}\, U^{(3)}_{31}) - 2A^*_{11}\, U^{(3)}_{31,YY} - 2(\partial^2/\partial Y^2)(V^{(3)}_{31}\, U^{(1)*}_{11}) \\
& - 4\mathrm{i}U^{(2)}_{11}\, U^{(2)}_{11,Y} + 2\alpha_1\, A_{11}\, U^{(2)}_{11,YY} - 2C^{(2)}_{11}\, U^{(2)}_{11,YY} \\
& - V^{(2)}_{20}\, U^{(2)}_{00,YY} + U^{(2)}_{00}\, V^{(2)}_{20,YY} - 4\mathrm{i}(\partial/\partial Y)(U^{(2)}_{22}\, U^{(2)}_{02}) - 2(\partial^2/\partial Y^2)(V^{(2)}_{02}\, U^{(2)}_{22}).
\end{aligned}$$

The kernels for §3 are given by

$$K_{11}^{(3a)} = -18T^3+6TT_0^2-8T_0^3+27T^2T_1-6TT_0T_1+8T_0^2T_1-12TT_1^2-7T_0T_1^2 +4T_1^3+27T^2T_2 -6TT_0T_2+10T_0^2T_2-24TT_1T_2+4T_0T_1T_2+7T_1^2T_2-12TT_2^2-9T_0T_2^2 +3T_1T_2^2+6T_2^3,$$

$$K_{11}^{(3b)} = -18T^3+18T^2T_0-6TT_0^2+4T_0^3+9T^2T_1-6TT_0T_1-11T_0^2T_1+16T_0T_1^2 -12T_1^3+27T^2T_2 -18TT_0T_2+5T_0^2T_2-12TT_1T_2-4T_0T_1T_2+20T_1^2T_2-12TT_2^2+6T_0T_2^2 -12T_1T_2^2+6T_2^3,$$

$$K_{11}^{(3c)} = -18T^3+18T^2T_0-6TT_0^2+4T_0^3+18T^2T_1-12TT_0T_1+5T_0^2T_1-6TT_1^2 +8T_0T_1^2+4T_1^3 +18T^2T_2-12TT_0T_2-11T_0^2T_2-12TT_1T_2-14T_0T_1T_2-14T_1^2T_2-6TT_2^2 +24T_0T_2^2+27T_1T_2^2-15T_2^3,$$

$$L_{11}^{(3a)} = -24TT_0+4T_0^2+12TT_1+13T_0T_1-8T_1^2+12TT_2+3T_0T_2-9T_1T_2-3T_2^2,$$

$$L_{11}^{(3b)} = 9TT_0-4T_0^2-21TT_1+5T_0T_1+8T_1^2+12TT_2-6T_0T_2-3T_2^2,$$

$$L_{11}^{(3c)} = 9TT_0-4T_0^2+9TT_1-6T_0T_1+4T_1^2-18TT_2+5T_0T_2-11T_1T_2+12T_2^2,$$

$$K_{31}^{(3)} = 27T^2-6TT_0-4T_0^2-18TT_1+12T_0T_1+4T_1^2-30TT_2+2T_0T_2-2T_1T_2+15T_2^2,$$

$$L_{31}^{(3a)} = 9T^3-69T^2T_0+54TT_0^2-12T_0^3-9T^2T_1+28TT_0T_1-12T_0^2T_1+10TT_1^2 -4T_0T_1^2-4T_1^3,$$

$$L_{31}^{(3b)} = 51T^2T_2+2TT_0T_2-6T_0^2T_2-30TT_1T_2+4T_0T_1T_2+6T_1^2T_2-37TT_2^2+3T_0T_2^2 +7T_1T_2^2+9T_2^3.$$

$$K_{20}^{(3)} = (-8T^3+15T^2T_0-10TT_0^2+2T_0^3+9T^2T_1-10TT_0T_1+4T_0^2T_1-4TT_1^2+T_0T_1^2 +T_1^3)/(T_0+T_1-2T),$$

$$K_{20}^{(4b)} = (T_0-T_2)(T_0-T_1)(-T_0+2T_1-T_2),$$

$$K_{20}^{(4b)} = -54T^4-30T^3T_0-9TT_0^3+82T^3T_1+30T^2T_0T_1+9TT_0^2T_1+T_0^3T_1-60T^2T_1^2 -29TT_0T_1^2-4T_0^2T_1^2+27TT_1^3+9T_0T_1^3-4T_1^4+82T^3T_2+30T^2T_0T_2+9TT_0^2T_2 +T_0^3T_2-78T^2T_1T_2-10TT_0T_1T_2-10T_0^2T_1T_2+34TT_1^2T_2 +21T_0T_1^2T_2-13T_1^3T_2 -60T^2T_2^2-29TT_0T_2^2-4T_0^2T_2^2+34TT_1T_2^2+21T_0T_1T_2^2-16T_1^2T_2^2 +27TT_2^3+9T_0T_2^3 -13T_1T_2^3-4T_2^4+82T^3T_3+30T^2T_0T_3+9TT_0^2T_3+7T_0^3T_3-78T^2T_1T_3 -10TT_0T_1T_3 +6T_0^2T_1T_3+34TT_1^2T_3-11T_0T_1^2T_3-7T_1^3T_3-78T^2T_2T_3-10TT_0T_1T_3 +6T_0^2T_2T_3 +30TT_1T_2T_3-54T_0T_1T_2T_3+16T_1^2T_2T_3+34TT_2^2T_3-11T_0T_2^2T_3 +16T_1T_2^2T_3-7T_2^3T_3 -60T^2T_3^2-29TT_0T_3^2-21T_0^2T_3^2+34TT_1T_3^2+37T_0T_1T_3^2-9T_1^2T_3^2+34TT_2T_3^2 +37T_0T_2T_3^2-20T_1T_2T_3^2-9T_2^2T_3^2+27TT_3^3-T_0T_3^3-11T_1T_3^3-11T_2T_3^3-T_3^4.$$

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